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LETTER TO THE EDITOR

Yang–Mills integrals

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Abstract
Two results are presented for reduced Yang–Mills (YM) integrals with different symmetry groups and dimensions: the first is a compact integral representation in terms of the relevant variables of the integral; the second is a method to analytically evaluate the integrals in cases of low order. This is exhibited by evaluating a YM integral over real symmetric matrices of order 3.

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1. Introduction

In recent years the complete reduction of the D-dimensional $SU(N)$ Yang–Mills (YM) theory was studied by many authors. The reduced supersymmetric partition function is an ordinary multiple integral of the form

$$Z = \int \prod_{\mu=1}^{D} \mathrm{d}X_\mu \,\mathrm{d}\Psi \,\mathrm{e}^{-S(X,\Psi)} \tag{1.1}$$

where the Euclidean action is

$$S(X,\Psi) = -\,\mathrm{Tr}\,[X_\mu, X_\nu][X_\mu, X_\nu] - \mathrm{Tr}\ \Psi\,\Gamma^\mu\,[X_\mu,\ \Psi] \tag{1.2}$$

and the matrix-valued gauge potentials X_μ and their fermionic superpartners Ψ have values in the $SU(N)$ Lie algebra

$$\begin{array}{ll} X_\mu = X_\mu^a T^a & \Psi = \Psi^a T^a \\ [T^a, T^b] = \mathrm{i} f^{abc} T_c & \mathrm{Tr}\ T^a T^b = \delta_{ab}. \end{array} \tag{1.3}$$

There are several motivations to investigate these integrals both in the maximally supersymmetric models, as well as in cases with less supersymmetry [1–4]. We refer to some recent papers [5, 6] for introduction to the subject, review of recent results and further references. The interest in understanding these integrals is such that the study of simpler models

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without fermionic partners and for a variety of groups has been considered very useful, also by means of extensive numerical investigations [7–11]. Unfortunately, a general rigorous method for computing equation (1.1) is still unknown. The development of techniques to understand these integrals for various values of N and D is very desirable, as well as techniques which allow the analytic evaluation of the integrals in cases of small order of the matrices.
In this letter we concentrate on the simpler, pure bosonic integrals of the form

$$Z = \int \prod_{\mu=1}^{D} \mathrm{d}X_\mu \; \exp(\mathrm{Tr}[X_\mu, X_\nu][X_\mu, X_\nu]). \tag{1.4}$$

The existence conditions and the convergence properties of (1.1) and (1.4) (and of the corresponding correlation functions) have been analytically established in the most general case and for any compact semi-simple gauge group recently in [6]. As it is well known, these models, besides the global unitary invariance for similarity transformation of the matrices, have a rotational $SO(D)$ vector invariance. The latter is the crucial ingredient for our two results: the first is a compact representation of equation (1.4) in terms of its natural variables. It holds for generic values of D and any group. The second result is a technique to analytically evaluate the integrals in the case of small-order matrices. We exploit it in the evaluation of a $5D$ integral corresponding to the YM integral over 3×3 real symmetric matrices. It seems likely that both results should be helpful also for the supersymmetric integrals.

It is useful to parametrize the random matrices with their independent entries. For instance we shall consider the ensemble of real traceless symmetric matrices of order 3:

$$X^\mu = \begin{pmatrix} a^\mu & u^\mu & v^\mu \\ u^\mu & -a^\mu - b^\mu & z^\mu \\ v^\mu & z^\mu & b^\mu \end{pmatrix} \qquad \mu = 1 \ldots D. \tag{1.5}$$

The action $S(X)$ is easily evaluated as a polynomial in the scalar products of the five independent D-dimensional vectors. The ensemble has the unitary invariance $SO(3)$ and we evaluate, for $D > 6$

$$\begin{aligned} Z_{SO(3)} &= \int \mathrm{d}^D a \, \mathrm{d}^D b \, \mathrm{d}^D u \, \mathrm{d}^D v \, \mathrm{d}^D z \; \mathrm{e}^{-S(X_\mu)} \\ &= \frac{\pi^{\frac{5D}{2}+1} \, 3^{1-D} \, 2^{6-\frac{7D}{2}}}{(D-4)(D-6)\,\Gamma\left(\frac{3D}{4} - \frac{1}{2}\right)\Gamma\left(\frac{D}{2} - \frac{1}{2}\right)}. \end{aligned} \tag{1.6}$$

This should not be confused with the ensemble of real anti-symmetric matrices, also having the same $SO(3)$ unitary invariance. The latter ensemble is parametrized by three D-dimensional vectors and corresponds to the only known integral which was exactly evaluated in every dimension both in the bosonic and supersymmetric cases [2, 5, 12]. Further evaluations were given for supersymmetric integrals in fixed dimension by the deformation technique [13], reproducing the conjecture of [14] and afterwards numerically confirmed for small gauge groups in [7].

2. A compact natural representation

Let us consider an ensemble of matrices X_μ, $\mu = 1, \ldots, D$, parametrized by a number n of D-dimensional real vectors. For instance, in the familiar ensemble of Hermitian traceless matrices of order N, invariant under the global $SU(N)$ group, there are $n = N^2 - 1$ real vectors. The action $S = -\mathrm{Tr}\,[X_\mu, X_\nu]^2$ is a homogeneous polynomial in the components of the vectors, and it only depends on the $n(n+1)/2$ scalar products of the n vectors, as it follows from the vector $SO(D)$ invariance of the model. The bosonic partition function is a multiple integral over nD variables, with an integrand which is a function of $n(n+1)/2$ variables. If $n \leqslant D$ the vectors are generically linearly independent and it seems desirable to express the partition function as an integral over the natural variables, that is the scalar products. This is expressed by the equation

$$Z = \int \prod_{j=1}^{n} \mathrm{d}\vec{v}_j \; F(B) = \int_R \prod_{i \leqslant j} \mathrm{d}b_{ij} \; F(B) I_{n,D}(B)$$
$$b_{ij} \equiv \vec{v}_i \cdot \vec{v}_j \equiv (v_i | v_j) = l_i l_j \, p_{ij} \qquad l_i \equiv \sqrt{(\vec{v}_i)^2}. \tag{2.1}$$

Here $F(B)$ stands for the usual Boltzmann weight (possibly multiplied by a Pfaffian or a determinant resulting from the integration over the fermionic fields), the region of integration R is described later in this section and $I_{n,D}(B)$ is the joint probability density for a set of n vectors $\vec{v}_i$, with given scalar products $b_{ij} = \vec{v}_i \cdot \vec{v}_j$ in the D-dimensional Euclidean space.

Proposition 1.

$$I_{n,D}(B) \equiv \int \prod_{i=1}^{n} \mathrm{d}\vec{v}_i \prod_{1 \leqslant j < k \leqslant n} \delta\left(b_{jk} - \vec{v}_j \cdot \vec{v}_k\right)$$
$$= [\det_n(B)]^{(D-n-1)/2} \left(\prod_{j=1}^{n} \frac{\Omega_{D-j}}{2} \right)$$

where

$$\Omega_{D-j} = 2 \frac{\pi^{(D-j+1)/2}}{\Gamma\left(\frac{D-j+1}{2}\right)} \qquad \det_n(B) = (l_1 l_2 \ldots l_n)^2 \det_n(P). \tag{2.2}$$

Proposition 1 may be proved from[5]*:*

Proposition 2. *For a given set of $n-1$ linearly independent vectors $\{\vec{v}_k\}$, $k = 1, \ldots, n-1$, in a real D-dimensional space, $n \leqslant D$,*

$$i_n = \int \mathrm{d}\vec{v}_n \, \delta\left(1 - \frac{\vec{v}_n^2}{l_n^2}\right) \delta\left(p_{1n} - \frac{\vec{v}_1 \cdot \vec{v}_n}{l_1 l_n}\right) \cdots \delta\left(p_{n-1,n} - \frac{\vec{v}_{n-1} \cdot \vec{v}_n}{l_{n-1} l_n}\right)$$
$$= \frac{\Omega_{D-n}}{2} l_n^D \frac{\det_n(P)^{(D-n-1)/2}}{\det_{n-1}(P)^{(D-n)/2}} = \frac{\Omega_{D-n}}{2} l_n^{n+1} \left(\prod_{j=1}^{n-1} l_j \right) \frac{\det_n(B)^{(D-n-1)/2}}{\det_{n-1}(B)^{(D-n)/2}} \tag{2.3}$$

where $\det_{n-1}(B)$ *is the leading principal minor obtained by deleting the last row and last column of B.*

We outline here a proof for proposition 2. The vector $\vec{v}_n$ may be decomposed on the basis of the $n-1$ external vectors and a $D-(n-1)$-dimensional vector $\vec{v}_{n\perp}$ in the orthogonal subspace:

$$\vec{v}_n = \omega_1 \vec{v}_1 + \cdots + \omega_{n-1} \vec{v}_{n-1} + \vec{v}_{n\perp}$$
$$\int \mathrm{d}\vec{v}_n = \sqrt{\det_{n-1}(B)} \int \mathrm{d}^{D-n+1} v_{n\perp} \left(\prod_{j=1}^{n-1} \int_{-\infty}^{\infty} \mathrm{d}\omega_j \right). \tag{2.4}$$

The vector $\vec{v}_{n\perp}$ only occurs in the first delta function $\delta(1 - \vec{v}_n^2 / l_n^2)$, and its integral is trivial

$$\int \mathrm{d}^{D-n+1} \vec{v}_{n\perp} \, \delta\left(1 - \frac{\vec{v}_{n\perp}^2 + \left(\sum \omega_j \vec{v}_j\right)^2}{l_n^2}\right) = \Omega_{D-n} \int_0^\infty \mathrm{d}\rho \, \rho^{D-n} \delta\left(1 - \frac{\rho^2 + \left(\sum \omega_j \vec{v}_j\right)^2}{l_n^2}\right)$$
$$= \frac{\Omega_{D-n}}{2} l_n^2 \left[l_n^2 - \left(\sum \omega_j \vec{v}_j \right)^2 \right]^{\frac{D-n-1}{2}}. \tag{2.5}$$

[5] A different proof of equation (2.2) may also be given by writing the integral representation for the delta functions and evaluating the integral of a random real symmetric matrix in external field.

The remaining $n-1$ delta functions fix the ω_j as the solution of the system $\sum_{j=1}^{n-1} b_{ij}\omega_j = l_i l_n\, p_{in} = b_{in}$, $i = 1, \ldots, n-1$, that is $\bar{\omega}_j = \sum_s [B^{-1}]_{js} b_{sn}$. Then

$$\begin{aligned} l_n^2 - \left(\sum \omega_j \vec{v}_j\right)^2 &= b_{nn} - \sum_{j,k=1}^{n-1} \bar{\omega}_j \bar{\omega}_k\, b_{jk} = b_{nn} - \sum_{j=1}^{n-1} \bar{\omega}_j\, b_{jn} \\ &= b_{nn} - \sum_{j,s=1}^{n-1} b_{ns}[B^{-1}]_{sj}\, b_{jn} = \frac{\det_n(B)}{\det_{n-1}(B)}. \end{aligned} \tag{2.6}$$

Finally proposition 2 is proved after using

$$\int_{-\infty}^{\infty} \prod_{j=1}^{n-1} \mathrm{d}\omega_j \prod_{k=1}^{n-1} \delta\left(p_{kn} - \frac{\sum_{r=1}^{n-1} \omega_r\, l_r\, p_{kr}}{l_n}\right) = \frac{l_n^{n-1}}{l_1 l_2 \ldots l_{n-1}} \frac{1}{\det_{n-1}(P)}. \tag{2.7}$$

Proposition 1 follows now from proposition 2 since the joint density $I_{n,D}(B)$, from its definition, may be written as the product of the integrals i_k.

The cosines p_{ij} are bound by $-1 \leqslant p_{ij} \leqslant 1$ but the derivation of equation (2.2) implies the stricter bound that $\det_k(P) > 0$ for each k, $k = 1, \ldots, n$, which is necessary and sufficient condition for the real symmetric matrix P to have all the eigenvalues positive. Equations (2.1), (2.2) provide a very simple reduction of the partition function to the smaller number of relevant variables which has general validity for different matrix ensembles and dimensions. It may be useful for numerical integration and possibly for more general purposes when averaging over random vectors[6]. The partition function expressed as multiple integral over the scalar products is closely related to the Ingham–Siegel integral [16][7]. In fact the well known YM integral for the $SU(2)$ invariant ensemble is a trivial case. After parametrization of the Hermitian traceless matrices of order 2 with 3 real entries, one obtains

$$\begin{aligned} Z_{SU(2)} &= \int \prod_{j=1}^{3} \mathrm{d}\vec{v}_j\ \mathrm{e}^{-S_{SU(2)}} \\ &= \frac{\pi^{3(D-1)/2}}{\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)\Gamma\left(\frac{D-2}{2}\right)} \int_{B>0} \mathrm{d}B\ \mathrm{e}^{-S_{SU(2)}}\ [\det(B)]^{(D-4)/2} \end{aligned}$$

where

$$S_{SU(2)} = 16(b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} - b_{12}^2 - b_{13}^2 - b_{23}^2) = 16\det(B)\ \mathrm{tr}\,(B^{-1}).$$

We change integration matrix variables $C_{ij} = (\det B)[B^{-1}]_{ij}$, $\mathrm{d}B = (\det C)^{-1}\,\mathrm{d}C$, $\det B = (\det C)^{1/2}$

$$\begin{aligned} \int_{B>0} \mathrm{d}B\ \mathrm{e}^{-S_{SU(2)}}\ [\det(B)]^{(D-4)/2} &= 2^{-3D} \int_{C>0} \mathrm{d}C\ (\det C)^{(D-8)/4}\ \mathrm{e}^{-\mathrm{tr}\, C} \\ &= \frac{\pi^{3/2}}{2^{3D}} \Gamma\left(\frac{D}{4}\right) \Gamma\left(\frac{D-1}{4}\right) \Gamma\left(\frac{D-2}{4}\right). \end{aligned} \tag{2.8}$$

Since for the $SU(2)$ case in $D = 10$ the Pfaffian arising from the fermion integration is proportional to $(\det B)^4$, also the supersymmetric $SU(2)$ integral is easily obtained, reproducing the result of [2, 12].

[6] See for instance chapter 21 of [15], where one studies averages over random orthogonal vectors. Our equation (2.2) for $n = 2$, $p_{12} = 0$, reproduces the result equation (21.1.11) of [15].

[7] The integrals of Ingham and Siegel have recently been discussed by Fyodorov [16].

3. Reduction by iterated projections

A useful technique which avoids the inconvenience of the Jacobian and positivity requirements is the *reduction by iterated projections*. The price is a more complicated expression of the action. To put it on general ground, let us consider the integral in n real vector variables of a function which only depends on the scalar products in D dimensions

$$I = \int \mathrm{d}^D x_1\, \mathrm{d}^D x_2 \ldots \mathrm{d}^D x_n F\big((x_i|x_j)_D; i \leqslant j\big) \tag{3.1}$$

Proposition 3. *Case $D \geqslant n$:*

$$I = \Omega_{D-1}\ldots\Omega_{D-n}\int_0^\infty \mathrm{d}x_1\, x_1^{D-1}\ldots\int_0^\infty \mathrm{d}x_n\, x_n^{D-n}\int_{-\infty}^\infty \prod_{i<j}\mathrm{d}x_{ij}\; F(\ldots) \tag{3.2}$$

$$(x_i|x_j)_D = x_{1i}x_{1j} + \cdots + x_{i-1,i}x_{i-1,j} + x_i x_{ij} \qquad i \leqslant j. \tag{3.3}$$

Case $D < n$:

$$I = \Omega_{D-1}\ldots\Omega_0\int_0^\infty \mathrm{d}x_1\, x_1^{D-1}\cdots\int_0^\infty \mathrm{d}x_D\int_{-\infty}^\infty \prod_{\substack{i<j\\ i\leqslant D}}\mathrm{d}x_{ij}\; F(\ldots) \tag{3.4}$$

$$\begin{aligned} (x_i|x_j)_D &= x_{1i}x_{1j} + \cdots + x_{Di}x_{Dj} \qquad && D < i \leqslant j \\ (x_i|x_j)_D &= x_{1i}x_{1j} + \cdots + x_i x_{ij} && i < j \quad i \leqslant D. \end{aligned} \tag{3.5}$$

Proof. The rotational invariance in the space R^D of all vectors allows us to choose the vector x_1 as the polar direction. Any other vector x_i is then described by its component parallel to x_1, denoted as x_{1i} living in R, and a vector orthogonal to it, which (with abuse of language) we again denote x_i, living in the space R^{D-1}. Thus we have the first sweep

$$I = \Omega_{D-1}\int_0^\infty \mathrm{d}x_1\, x_1^{D-1}\int_{-\infty}^\infty \mathrm{d}x_{12}\ldots\mathrm{d}x_{1n}\int \mathrm{d}x_2^{D-1}\ldots\mathrm{d}x_n^{D-1}F(\ldots) \tag{3.6}$$

where the scalar products in F are easily evaluated:

$$(x_1|x_1)_D = x_1^2 \qquad (x_1|x_i)_D = x_1 x_{1i}$$
$$(x_i|x_j)_D = x_{1i}x_{1j} + (x_i|x_j)_{D-1} \qquad 1 < i \leqslant j.$$

Next we choose the vector x_2 as the polar direction in R^{D-1}, and introduce accordingly new real variables x_{2i} and vectors x_i in R^{D-2}, $i = 3, \ldots, n$. The integral is now

$$\begin{aligned} I = \Omega_{D-1}\Omega_{D-2}&\int_0^\infty \mathrm{d}x_1\, x_1^{D-1}\int_0^\infty \mathrm{d}x_2\, x_2^{D-2}\int_{-\infty}^\infty \mathrm{d}x_{12}\ldots\mathrm{d}x_{1n}\,\mathrm{d}x_{23}\ldots\,\mathrm{d}x_{2n} \\ &\times\int \mathrm{d}^{D-2}x_3\ldots\mathrm{d}^{D-2}x_n\; F(\ldots). \end{aligned} \tag{3.7}$$

The scalar products in R^{D-1} that still appear in F are evaluated:

$$(x_2|x_2)_{D-1} = x_2^2 \qquad (x_2|x_i)_{D-1} = x_2 x_{2i}$$
$$(x_i|x_j)_{D-1} = x_{2i}x_{2j} + (x_i|x_j)_{D-2} \qquad 2 < i \leqslant j.$$

The process is iterated and we have two situations: $D \geqslant n$, $D < n$. In the first case we can iterate the process n times, to obtain an integral in n positive variables $x_1 \ldots x_n$ and $\frac{1}{2}n(n-1)$ real variables x_{ij}, $i < j$. When $D < n$ the process can be iterated only D times; we obtain D positive variables $x_1 \ldots x_D$, and $\frac{1}{2}D(2n - D - 1)$ real variables x_{ij}, $i < j$, $i = 1 \ldots D$. □

4. The YM integral with 3 × 3 real symmetric matrices

The 3×3 traceless real symmetric matrices of the YM integral are parametrized as indicated in equation (1.5). The partition function

$$Z_{SO(3)} = \int \mathrm{d}^D a\, \mathrm{d}^D b\, \mathrm{d}^D u\, \mathrm{d}^D v\, \mathrm{d}^D z\, \mathrm{e}^{4S_0} \tag{4.1}$$

contains the following expression for the action, in terms of scalar products such as $(a|b)_D = \sum_\mu a^\mu b^\mu$:

$$\begin{aligned} S_0 = -4[a^2u^2 - (a|u)^2] - [a^2z^2 - (a|z)^2] - [a^2v^2 - (a|v)^2] - [b^2u^2 - (b|u)^2] \\ -4[b^2z^2 - (b|z)^2] - [b^2v^2 - (b|v)^2] - [u^2z^2 - (u|z)^2] - [v^2z^2 - (v|z)^2] \\ -[u^2v^2 - (u|v)^2] - 4[u^2(a|b) - (u|a)(u|b)] - 4[z^2(a|b) - (z|a)(z|b)] \\ +2[v^2(a|b) - (v|a)(v|b)] + 6(a|z)(u|v) - 6(a|v)(u|z) \\ +6(b|u)(v|z) - 6(b|v)(u|z). \end{aligned} \tag{4.2}$$

The huge invariance of the action can be exploited by reducing to a smaller number of effective variables. We have been able to compute $Z_{SO(3)}$ by using the technique of successive projections, described in the previous section.

For $D \geqslant 5$ the reduction yields a 15-dimensional integral:

$$\begin{aligned} Z_{SO(3)} = \Omega_{D-1} \ldots \Omega_{D-5} \int_0^\infty \mathrm{d}v\, v^{D-1} \int_0^\infty \mathrm{d}z\, z^{D-2} \int_0^\infty \mathrm{d}u\, u^{D-3} \int_0^\infty \mathrm{d}a\, a^{D-4} \int_0^\infty \mathrm{d}b\, b^{D-5} \\ \times \int_{-\infty}^{\infty} \mathrm{d}z_v\, \mathrm{d}u_v\, \mathrm{d}a_v\, \mathrm{d}b_v\, \mathrm{d}u_z\, \mathrm{d}a_z\, \mathrm{d}b_z\, \mathrm{d}a_u\, \mathrm{d}b_u\, \mathrm{d}b_a\ \exp(S_1 + S_2 + S_3 + S_4) \end{aligned} \tag{4.3}$$

where the action in the reduced variables has been decomposed in terms with the following meaning. S_1 includes all terms in the variables a, b_a and b, that terminated the reduction procedure and can be integrated immediately:

$$\begin{aligned} S_1 = -4a^2[v^2 + (z^2 + z_v^2) + 4(u^2 + u_v^2 + u_z^2)] - 4(b^2 + b_a^2)[v^2 + 4(z^2 + z_v^2) + (u^2 + u_v^2 + u_z^2)] \\ +8ab_a[v^2 - 2(z^2 + z_v^2 + u^2 + u_v^2 + u_z^2)]. \end{aligned} \tag{4.4}$$

The integration yields a factor that does not depend on the six variables a_x, b_x, where $x = u, v, z$. These variables appear in the action quadratically and linearly. The quadratic terms are collected in S_2 and the linear terms in S_3, in a form suitable for a Gaussian integration:

$$S_2 = -4X^t M X \qquad S_3 = 8vX^t Y \tag{4.5}$$

where

$$X = \begin{pmatrix} a_v + 2b_v \\ 2a_v + b_v \\ a_z + 2b_z \\ 2a_z + b_z \\ a_u + 2b_u \\ 2a_u + b_u \end{pmatrix} \qquad Y = \begin{pmatrix} -zu_z \\ -zu_z \\ 2u_z z_v - zu_v \\ -u_z z_v + 2zu_v \\ 2uz_v \\ -uz_v \end{pmatrix}$$

$$M = \begin{pmatrix} z^2 & 0 & -zz_v & 0 & 0 & 0 \\ 0 & u^2 + u_z^2 & 0 & -u_z u_v & 0 & -uu_v \\ -zz_v & 0 & v^2 + z_v^2 & -v^2 & 0 & 0 \\ 0 & -u_z u_v & -v^2 & v^2 + u^2 + u_v^2 & 0 & -uu_z \\ 0 & 0 & 0 & 0 & v^2 + z^2 + z_v^2 & -v^2 \\ 0 & -uu_v & 0 & -uu_z & -v^2 & v^2 + u_v^2 + u_z^2 \end{pmatrix}.$$

The remaining part S_4 of the action contains terms in the reduced variables originating from the off-diagonal matrix elements, namely v, z, z_v, u, u_v and u_z:

$$S_4 = -4z^2(v^2 + u_v^2) - 4u_z^2(v^2 + z_v^2) - 4u^2(v^2 + z^2 + z_v^2) + 8zu_zu_vz_v. \tag{4.6}$$

As we described, the $Z_{SO(3)}$ integral can be done in independent steps. We have a factor coming from the triple integral

$$\begin{aligned} I_1 &= \int_0^\infty \mathrm{d}a\, a^{D-4} \int_0^\infty b^{D-5}\, \mathrm{d}b \int_{-\infty}^\infty \mathrm{d}b_a\, \mathrm{e}^{S_1} \\ &= \frac{\sqrt{\pi}}{4}(12)^{-D+3} \frac{\Gamma\left(\frac{D-3}{2}\right)\Gamma\left(\frac{D-4}{2}\right)}{[v^2(u^2+u_v^2+u_z^2+z^2+z_v^2)+(z^2+z_v^2)(u^2+u_v^2+u_z^2)]^{(D-3)/2}}. \end{aligned} \tag{4.7}$$

At the same time one can do the Gaussian integral

$$\begin{aligned} I_2 &= \int_{-\infty}^\infty \mathrm{d}a_v\, \mathrm{d}b_v\, \mathrm{d}a_z\, \mathrm{d}b_z\, \mathrm{d}a_u\, \mathrm{d}b_u\, \mathrm{e}^{S_2+S_3} \\ &= \frac{1}{12^3}\int \mathrm{d}^6X\, \mathrm{e}^{-X^tMX+4vX^tY} = \frac{\pi^3}{12^3}(\det M)^{-1/2}\, \mathrm{e}^{S_5}. \end{aligned} \tag{4.8}$$

The computations for inverting the matrix M and evaluating the quadratic form $S_5 = 4v^2Y^tM^{-1}Y$ were done with the aid of Mathematica:

$$S_5 = 4(v^2+z_v^2)(u^2+u_z^2) + 4z^2(v^2+u_v^2) - 8zz_vu_vu_z - 4\frac{(vzu)^2}{z^2+z_v^2} - 4\frac{(vzu)^2}{u^2+u_v^2+u_z^2}$$

$$\det M = v^4z^2u^2(z^2+z_v^2)(u^2+u_v^2+u_z^2).$$

In summing the expressions S_4 and S_5, to obtain the residual action, many terms cancel:

$$S_4 + S_5 = -4z^2u^2 - 4v^2z^2u^2\frac{u^2+u_v^2+u_z^2+z^2+z_v^2}{(z^2+z_v^2)(u^2+u_v^2+u_z^2)}. \tag{4.9}$$

We arrive at the following stage of computation of the full integral:

$$Z_{SO(3)} = \frac{1}{4}\pi^{7/2}12^{-D}\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(\frac{D}{2}-\frac{3}{2}\right)\Omega_{D-1}\ldots\Omega_{D-5}\, I_3 \tag{4.10}$$

where we still have to evaluate

$$\begin{aligned} I_3 = \int_0^\infty \mathrm{d}v\, v^{D-3}\int_0^\infty \mathrm{d}z\, z^{D-3}\int_0^\infty \mathrm{d}u\, u^{D-4}\int_{-\infty}^\infty \mathrm{d}z_v\, \mathrm{d}u_v\, \mathrm{d}u_z\, \mathrm{e}^{S_4+S_5} \\ \times[v^2(u^2+u_v^2+u_z^2+z^2+z_v^2)+(z^2+z_v^2)(u^2+u_v^2+u_z^2)]^{-\frac{1}{2}(D-3)} \\ \times[(u^2+u_z^2+u_v^2)(z^2+z_v^2)]^{-1/2}. \end{aligned}$$

Since the variables u_v and u_z always appear in the combination $u_v^2 + u_z^2 = \rho^2$, we introduce polar variables obtaining a factor 2π from angle integration. It is also convenient to change from the variable v to the variable x:

$$x = v\sqrt{\frac{u^2+\rho^2+z^2+z_v^2}{(u^2+\rho^2)(z^2+z_v^2)}}$$

$$\begin{aligned} I_3 = 2\pi\int_0^\infty \mathrm{d}x\, x^{D-3}(1+x^2)^{-\frac{D-3}{2}}\int_0^\infty \mathrm{d}z\, z^{D-3}\int_0^\infty \mathrm{d}u\, u^{D-4}\exp(-4u^2z^2(1+x^2) \\ \times\int_0^\infty \rho\, \mathrm{d}\rho\int_{-\infty}^\infty \mathrm{d}z_v\, (u^2+\rho^2+z^2+z_v^2)^{-\frac{D}{2}+1}. \end{aligned} \tag{4.11}$$

The double integral in ρ and z_v is elementary, and poses the restriction $D > 5$ for convergence,

$$\int_{-\infty}^{\infty} \mathrm{d}z_v \int_0^{\infty} \rho\, \mathrm{d}\rho\, (u^2+\rho^2+z^2+z_v^2)^{-\frac{D}{2}+1} = (z^2+u^2)^{-\frac{D}{2}+\frac{5}{2}} B\left(\frac{D-5}{2}, \frac{3}{2}\right).$$

The remaining integrals are easily done with the substitution $u = (1+x^2)^{-1/4} r\cos\theta$ and $z = (1+x^2)^{-1/4} r \sin\theta$. We obtain, with the further restriction $D > 6$ for convergence,

$$I_3 = 2\pi B\left(\frac{D-5}{2}, \frac{3}{2}\right) \int_0^{\infty} \mathrm{d}x\, x^{D-3} (1+x^2)^{-\frac{3}{4}D+\frac{3}{2}} \int_0^{\pi/2} \mathrm{d}\theta \cos^{D-4}\theta \sin^{D-3}\theta$$
$$\times \int_0^{\infty} \mathrm{d}r\, r^{D-1} \exp(-4r^4 \sin^2\theta\cos^2\theta) = 4\pi^{5/2} 2^{-\frac{3}{2}D} \frac{\Gamma\left(\frac{D}{2}-1\right)\Gamma\left(\frac{D}{2}-3\right)}{\Gamma\left(\frac{3}{4}D-\frac{3}{2}\right)}. \tag{4.12}$$

The final result, for $D > 6$ is

$$Z_{SO(3)} = \frac{\pi^{\frac{5D}{2}+1}\, 3^{1-D}\, 2^{6-\frac{7D}{2}}}{(D-4)(D-6)\,\Gamma\left(\frac{3D}{4}-\frac{1}{2}\right)\Gamma\left(\frac{D}{2}-\frac{1}{2}\right)}. \tag{4.13}$$

In the case $D = 4$, one should truncate the reduction by projections, as described in the previous section (since the five vectors are not linearly independent). This produces a set of 14 integration variables and an action which, in simple terms, are obtained from equation (4.3) by suppressing the integral in b and all terms containing b in S_1. We checked that also in this case the integrations lead to a divergent result.

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